Descriptive Set Theory HW 3

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Problem 1. Let X be a Polish space.

1. Show that if $K \subseteq X$ is countable and compact, then its Cantor-Bendixson rank $|K|_C$ is not a limit ordinal.
2. For each non-limit ordinal $\alpha < \omega_1$, construct a countable compact subset K_α of $\mathcal{C}$, whose Cantor-Bendixson rank is exactly α.

Solution.

1. Let α denote the Cantor-Bendixson rank of K. Since K is a countable Polish space (it's compact), we have that α is the first ordinal for which $K^\alpha = \varnothing$. To show that α isn't a limit ordinal, it's enough to show that, if $\lambda < \omega_1$ is a limit ordinal and $K^\beta \neq \varnothing$ for each $\beta < \lambda$, then $K^\lambda \neq \varnothing$. Towards this end, fix a cofinal ω-sequence $\lambda_n \to \lambda$, and choose $x_n \in K^{\lambda_n} \subseteq K$. Since K is compact, some subsequence converges to an element x. Abusing notation, call this subsequence x_n. Since the K^β's form a decreasing chain, and $(\lambda_n)_{n<\omega}$ is a cofinal sequence, it follows that $x \in \bigcap_{\beta<\lambda} K^\beta$, as each K^β is closed.

2. **Claim 1.** For each $\alpha < \omega_1$, $\omega^\alpha + 1$ is a countable compact set with Cantor-Bendixson rank $\alpha + 1$, viewed as a topological space with the order topology.

   Proof. It's well known that countable successor ordinals are compact with the order topology, so it's enough to argue by induction that the α-iterated Cantor-Bendixon derivative of $\omega^\alpha + 1$ is $\{\omega^\alpha\}$, which implies that the Cantor-Bendixson rank is $\alpha + 1$. The base case $\alpha = 0$ is clear, so assume that $\alpha = \beta + 1$.

   Then $\omega^\alpha + 1 = [0, \omega^\beta\omega] = [0, \omega^\beta] \cup \bigcup_n [\omega^\beta n + 1, \omega^\beta(n+1)] \cup \{\omega^\alpha\}$, where each closed interval is homeomorphic to a copy of $[0, \omega^\beta]$. By induction, we have that the β-iterated Cantor-Bendixon derivative of $\omega^\alpha + 1$ is $\{\omega^\beta n \colon n > 0\} \cup \{\omega^\alpha\}$, implying that the α-iterated Cantor-Bendixon derivative of $\omega^\alpha + 1$ is $\{\omega^\alpha\}$.

For the limit case, fix a limit ordinal λ, and $\xi < \lambda$. By induction, we have that ω^α is in the ξ-iterated Cantor-Bendixon derivative of $\omega^\lambda + 1$, for any $\alpha \in \lambda - \xi$. This implies that ω^λ is then in the ξ-iterated Cantor-Bendixon derivative of $\omega^\lambda + 1$. Since ξ was arbitrary, we have that ω^λ is then in the λ-iterated Cantor-Bendixon derivative of $\omega^\lambda + 1$. Conversely, if $\beta < \omega^\alpha < \omega^\lambda$, then the induction hypothesis implies that β isn't in the α-iterated Cantor-Bendixon derivative of $\omega^\lambda + 1$. So, β isn't in the λ-iterated Cantor-Bendixon derivative of $\omega^\lambda + 1$. This completes the induction, and the claim.

$\square$

Now, notice that we can identify $\mathbb{Q}$ in the Cantor set with the set of all eventually 0 sequences that aren't constant sequences, so it's enough to find countable compact subsets of $\mathbb{Q}$ with the desired Cantor-Bendixson rank. But, for each $\alpha < \omega_1$, we may think of $\omega^\alpha + 1$ as a countable compact subset of $\mathbb{Q}$. This is because $\mathbb{Q}$ contains homeomorphic copies of all countable ordinals with the order topology. The result follows.

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Problem 2. Let X be a second countable zero-dimensional space.

1. Prove Kuratowskis reduction property: If $A, B \subseteq X$ are open, there are open $A^* \subseteq A, B^* \subseteq B$ with $A^* \cup B^* = A \cup B$ and $A^* \cap B^* = \varnothing$.

2. Conclude the following separation property: For any disjoint closed sets $A, B \subseteq X$, there is a clopen set C separating A and B, i.e. $A \subseteq C$ and $B \cap C = \varnothing$.

Solution.

1. Write $A = \bigcup_n A_n, B = \bigcup_n B_n$, where each A_n, B_n are clopen. For each $x \in A \cup B$, let $a(x)$ be the least n such that $x \in A_n$ if such exists, and ∞ otherwise. Define $b(x)$ in the similar way. Partition $A \cup B$ into A^* and B^* as follows:

$$x \in A^* \Leftrightarrow a(x) \leq b(x) \text{ and } x \in B^* \Leftrightarrow b(x) < a(x).$$

It's not hard to see that $A^* \cup B^* = A \cup B$ and $A^* \cap B^* = \varnothing$. Without too much loss of generality, it's enough to show that A^* is open. If $x \in A^*$, then $a(x) \leq b(x)$ and $a(x)$ is finite. So, for each $k < a(x)$, $x \in (A_k \cup B_k)^c$, which is clopen by choice of A_k, B_k. Denote $(A_k \cup B_k)^c$ by V_k. Then $x \in A_{a(x)} \cap \bigcap_{k<a(x)} V_k \subseteq A^*$. So A^* is open as desired. Showing B^* is open is similar, except you also intersect with $A^c_{b(x)}$ to ensure that $b(x) < a(x)$.

2. By part 1, there's open $A^* \subseteq A^c$, $B^* \subseteq B^c$ with $A^* \cup B^* = A^c \cup B^c = X$ and $A^* \cap B^* = \varnothing$. Let $C = (A^*)^c = B^*$, which is clopen. By choice of C, we have that $A \subseteq C$ and $B \cap C = \varnothing$.

$\star$

Problem 3. Let X be a nonempty zero-dimensional Polish space such that all of its compact subsets have empty interior. Fix a complete compatible metric and (a) prove that there is a Luzin scheme $(A_s)_{s \in \omega^{<\omega}}$ with vanishing diameter and satisfying the following properties:

1. $A_\varnothing = X$;

2. A_s is nonempty and clopen;

3. $A_s = \bigcup_{i<\omega} A_{s^\frown i}$.

From this, (b) derive the Alexandrov-Urysohn theorem, i.e. show that the Baire space is the only topological space, up to homeomorphism, that satisfies the hypothesis above.

Solution. We do part (b) first. Observe that the standard basis for ω^ω witnesses that it is zero-dimensional and Polish. Further, compact subsets of ω^ω are the branches of finitely branching trees on ω, and so therefore cannot contain any N_s for $s \in \omega^{<\omega}$. Before we start the Luzin scheme construction, we note a useful fact.

Claim 2. Given an open U, and $\varepsilon > 0$, there are disjoint clopen V_i with $\operatorname{diam} V_i < \varepsilon$ such that $U = \bigcup_i V_i$.

Proof. Let D be a countable dense set. First, notice that we may write $U = \bigcup_i W_i$ where each W_i is clopen, $\operatorname{diam} W_i < \varepsilon$, but the W_i's aren't necessarily disjoint. To see this, cover U with balls with centers $d \in D \cap U$ with sufficiently small radius, then write those balls as unions of clopen sets (which we can do because our space is zero-dimensional).

Then, given $U = \bigcup_i W_i$, define $V_i = W_i - \bigcup_{k<i} W_k$. Then, the V_i's are disjoint by construction, each is clopen because each W_i is clopen, $U = \bigcup_i V_i$, and $\mathrm{diam} V_i \leq \mathrm{diam} W_i < \varepsilon$. $\square$

Now, we construct the Luzin scheme as follows: Set $A_\varnothing = X$. Now, assume that we have defined A_s, nonempty and clopen. By closed-ness, we have that $\overline{A_s} = A_s$. By open-ness, it then follows that A_s isn't compact as compact subsets have empty interior.

In particular, we have that A_s isn't totally bounded. So, we can find some $\delta < 1/|s|$ such that there's no finite covering of A_s by open sets with diameter at most δ. By claim 2, we have that $A_s = \bigcup_i V_i$ with $\mathrm{diam} V_i < \delta$. But, since δ witnesses the failure of total boundedness, we may just assume that each V_i is nonempty. Then, put $A_{s^\frown i} = V_i$. This completes the construction.

By construction, the A_s have vanishing diameter, so the associated Luzin scheme map is a homeomorphism between X and ω^ω (it satisfies enough of the right conditions of Prop 5.4 in Anush's notes).

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Problem 4. Let X, Y be topological spaces. We say that a continuous function $f \colon X \to Y$ is category-preserving if preimages of meager sets are meager.

1. Show that any continuous open map $f \colon X \to Y$ is category-preserving (in fact, preimages of nowhere dense are nowhere dense). In particular, projections are category-preserving.

2. For topological spaces X, Y, if X is Baire, then, for a continuous map $f \colon X \to Y$, the following are equivalent:

   (a) f is category preserving.

   (b) f-preimages of nowhere dense sets are nowhere dense.

   (c) f-preimages of dense open sets are dense.

Solution.

1. Let $N \subseteq Y$ be nowhere dense and let $U \subseteq X$ be nonempty and open. Since f is open, so too is $f"U$. Since N is nowhere dense, there's a nonempty $W \subseteq f"U$ such that $W \cap N = \varnothing$. If we set $V = f^{-1}[W] \cap U$, then V is nonempty and $V \cap f^{-1}[N] = \varnothing$.

2. First notice that (b) is equivalent to preimages of closed nowhere dense sets being closed nowhere dense, because the closure of a nowhere dense

set is nowhere dense. This implies that (b) and (c) are equivalent as complements of closed nowhere dense sets are open dense. It is straight forward to check that (b) implies (a), and so we show that (a) implies (b).

Towards this end, let N be closed nowhere dense. Then $f^{-1}[N]$ is closed and meager by (a). If $f^{-1}[N]$ weren't nowhere dense, then it has nonempty interior, so we could fix a nonempty open $U \subseteq f^{-1}[N]$. But then U is an open meager set, contradicting that X is Baire.

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Problem 5. Give an example of a function that is continuous at every irrational but discontinuous at every rational. However, prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every rational but discontinuous at every irrational.

Solution. The standard example of a function that is continuous only at irrationals is Thomae's function, where $x \mapsto 0$ for irrational x and $a/b \mapsto 1/b$, where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $\gcd(a, b) = 1$.

Now, let $f : \mathbb{R} \to \mathbb{R}$ be a map that is continuous at every rational. We show that it's also continuous at an irrational. To see this, let $\mathfrak{C}$ be the set of all x such that f is continuous at x.

Claim 3. $\mathfrak{C}$ is G_δ.

Proof. First observe that slight massaging of the definition of continuity gives us

$$x \in \mathfrak{C} \iff (\forall q \in \mathbb{Q}^+)(\exists r \in \mathbb{Q}^+)(\forall a)(\forall b)(a, b \in B(x, r) \to |f(a) - f(b)| < q).$$

For a rational q, if we let U_q be the set of all x that satisfy the scope of the quantifier $\forall q$, then this set is open. Indeed, if $x \in U_q$ and r_x witnesses this, then $B(x, r_x) \subseteq U_q$, as given $y \in B(x, r_x)$, choose r_y small enough so that $B(y, r_y) \subseteq B(x, r_x)$. Then, for any $a, b \in B(y, r_y)$, we have $a, b \in B(x, r_x)$, implying that $|f(a) - f(b)| < q$. So, $\mathfrak{C} = \bigcap_n U_n$ and we win. $\square$

If f is continuous at each rational, then we must have that $\mathbb{Q} \subsetneq \mathfrak{C}$, as we know that $\mathbb{Q}$ isn't a G_δ set from a previous exercise. $\star$

Problem 6. Prove that finite bounded games are determined.

Solution. Let the tree $T \subseteq A^{<n}$, for some n, with payoff set $D \subseteq \text{Leaves}(T)$. By extending the tree to T^*, may assume that each play has exactly length n. To do this, though, we also form a new payoff set D^* by replacing any $d \in D$ that was extended by each new element $x \in \text{Leaves}(T^*)$ that now end extends d. Then, the game (T^*, D^*) is determined iff (T, D) is determined. Indeed, following the same winning strategy of one of the games would also be a winning strategy for another. Following the hint, assume without loss of generality that n is even. If $\exists a_1 \forall a_2 \ldots \exists a_{n1} \forall a_n ((a_1, \ldots, a_n) \in D)$, then player I has a winning strategy. Otherwise, $\forall a_1 \exists a_2 \ldots \forall a_{n1} \exists a_n ((a_1, \ldots, a_n) \notin D)$, implying that II has a winning strategy. $\star$

Problem 7. Prove the determinacy of finite games, and then conclude the determinacy of clopen infinite games.

Solution. Assume that player II doesn't have a winning strategy for T with payoff set D. We describe one for player I. Call a position $s \in T$ determined for II, if either s is a terminal node and $s \notin D$, or from that point on, player II has a winning strategy. Otherwise, say s is undetermined for II. One way to think of s being determined for II is that if the board position were set up to have starting position $s \in T$, then II knows how to win.

Claim 4. If s is undetermined for player II and it's player I's move, then $s^\frown a$ is undetermined for II for some $a \in A$; that is, player I can play a non losing move. In particular, if player I plays a terminal node, it must be in D.

Proof. If every $s^\frown a$ was determined for II, then s would also be determined for II, because regardless of player I's move, II knows how to play to win. $\square$

Claim 5. If s is undetermined for player II and it's player II's move, then $s^\frown a$ is undetermined for II for every $a \in A$. In particular, if player II plays a terminal node, it must be in D.

Proof. If $s^\frown a$ were determined for II, then at position s, if player II plays a and then follows his winning strategy for $s^\frown a$, player II would win any board position from s. So, s is determined for II. $\square$

Consider a strategy for player I that goes as follows: at each of his turns, player I appeals to claim 4 to play some non losing move. By claim 5, every such response by II is undetermined for II, and so on player I's next turn, he uses claim 4 again to continue playing a non losing move. Call this strategy σ. We claim this is winning for player I. To see this, observe that by induction one can show that if the play terminates at any position x, then $x \in D$. Since

T has no infinite branches, each play using σ must terminate, and therefore $[\sigma] \subseteq D$. So σ is winning for player I.

Regarding the second part of the question, if our tree is $A^{<\omega}$ and our payoff set $D \subseteq A^\omega$ is clopen, then for each $x \in A^\omega$, there's a least $n_x < \omega$ such that $N_{n_x} \subseteq D$ or $N_{n_x} \subseteq D^c$. In other words, we know at some finite stage whether or not x is in our payoff set D. Consider the tree T whose terminal nodes are $\{x \restriction n_x \colon x \in A^\omega\}$, and define the payoff set $D^* = \{x \restriction n_x \colon x \in D\}$. This is a finite game, and so the above implies that it is determined. Then, a winning strategy for (T, D^*) for any player will correspond to a winning strategy in $(A^{<\omega}, D)$ for that same player. $\star$

Problem 8. Let X be a second countable Baire space. Show that the σ-ideal $\mathrm{MGR}(X)$ has the countable chain condition in $\mathrm{BP}(X)$, i.e. there is no uncountable family $\mathcal{A} \subseteq \mathrm{BP}(X)$ of non-meager sets such that for any two distinct $A, B \in \mathcal{A}$, $A \cap B$ is meager.

Solution. Assume instead that there is an uncountable family $\mathcal{A} \subseteq \mathrm{BP}(X)$ of non-meager sets such that for any two distinct $A, B \in \mathcal{A}$, $A \cap B$ is meager. For each $A \in \mathcal{A}$, let U_A be an open set such that $A =^* U_A$. Note that since each A is non-meager, U_A is nonempty. Given distinct $A, B \in \mathcal{A}$, note that $U_A \cap U_B$ is meager, as $U_A \cap U_B \subseteq (U_A - A) \cup (A \cap B) \cup (U_B - B)$, and each of these things is meager. This forces that each $U_A \cap U_B = \varnothing$, as nonempty open subsets of a Baire space are non-meager. Then $\{U_A \colon A \in \mathcal{A}\}$ is an uncountable disjoint collection of open sets, contradicting that X is second countable. $\star$

Problem 9. Let X be a topological space.

1. If $A_n \subseteq X$, then for any $U \subseteq X$
$$U \Vdash \bigcap_n A_n \iff (\forall n) U \Vdash A_n.$$

2. If X is a Baire space, A has the BP in X and $U \subseteq X$ is nonempty open, then
$$U \Vdash A^c \iff (\forall V \subseteq U) V \nVdash A,$$
as V varies over a weak basis.

3. If X is a Baire space, the sets $A_n \subseteq X$ have the BP, and U is nonempty open, then
$$U \Vdash \bigcup_n A_n \iff (\forall V \subseteq U)(\exists W \subseteq V)(\exists n) W \Vdash A_n,$$
as V and W vary over a weak basis.

Note: By Anush's definition of a weak basis, the V and W are all nonempty.

Solution.

1. If $A_n \subseteq X$ and $U \subseteq X$, then the $\Rightarrow$ direction is clear because $\bigcap_n A_n \subseteq A_n$. Going the other direction, since $U \Vdash A_n$, we have that $U - A_n$ is meager for each n. Then, $U - \bigcap_n A_n = \bigcup_n (U - A_n)$ is meager as well. So, $U \Vdash \bigcup_n A_n$.

2. $\Rightarrow$: Assume instead that $U \Vdash A^c$ but for some nonempty $V \subseteq U$, $V \Vdash A$. Since $V \subseteq U$, we have as well that $V \Vdash A$. By (1), $V \Vdash \varnothing$, implying that V is meager, contradicting that X is Baire.

   $\Leftarrow$: If $U \nVdash A^c$, then $U - A^c = U \cap A$ is non-meager. Now, $U \cap A$ has the BP, and so the Baire Alternative implies that $U \cap A$ is comeager in some nonempty open set W. That is, $W \Vdash U \cap A$. Since W is nonempty, we must have that $W \cap U \cap A$ is nonempty as well (lest $W \Vdash \varnothing$). Now, properties of the forcing relation imply that $W \cap U \Vdash A$. Since $W \cap U \cap A$ is nonempty, so too is $W \cap U$. By choosing any $V \subseteq W \cap U$ in the weak basis, we have $V \Vdash A$ as desired.

3. $\Rightarrow$: Assume $U \Vdash \bigcup_n A_n$ but the result fails. Without loss of generality, we may assume that for each nonempty subset $W \subseteq U$ and for all n, $W \nVdash A_n$. By (2), we have that $U \Vdash A_n^c$ for each n, implying by (1) that $U \Vdash (\bigcup_n A_n)^c$. Since X is Baire, this contradicts that $U \Vdash \bigcup_n A_n$.

   $\Leftarrow$: If $U \nVdash \bigcup_n A_n$, then by (2), we have $V \Vdash \bigcap_n A_n^c$ for some nonempty $V \subseteq U$, since $\bigcup_n A_n$ has BP. By (1), this is equivalent to $V \Vdash A_n^c$ for each n. Then, for any nonempty $W \subseteq V$ and any n, using (2) again gives us that $W \nVdash A_n$. Then the RHS of the $\Longleftrightarrow$ is false, yielding the result.

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